The Euler line: Students' Exploration in a Dynamic Geometry Environment

Thomas Lingefjard
e-mail: Thomas.Lingefjard@gu.se
Department of Mathematics Education
University of Gothenburg,
Sweden

Jonaki B Ghosh
e-mail: jonakibghosh@gmail.com
Department of Elementary Education
Lady Shri Ram College for Women
University of Delhi,
New Delhi, India

Abstract

This study investigates the impact of a dynamic geometry environment, GeoGebra, in enabling grade 9 students to explore a geometrical task related to the Euler's line. The research subjects were grade 9 students in India and Sweden who had no prior experience of using dynamic geometry. Their approach to geometrical reasoning, argumentation and conjecture making has been analysed using Marton's theory of variation, Pea's theory of technology as amplifier and reorganizer and with Marrades and Gutiérrez categorization of proofs. Results indicate that when students of both countries worked on geometrical tasks in GeoGebra, they arrived at similar conclusions. For both groups of students, GeoGebra played the role of amplifier and reorganiser in enabling their explorations. Further, in some of the construction and dragging episodes it was observed that the four functions of variation, namely, contrast, generalisation, separation and fusion played a significant role in developing students' geometrical thinking. Results further indicate that the dynamic geometry environment (DGE) facilitated the making of conjectures and developing empirical proof. However, when it came to developing a deductive proof, students required adequate scaffolding. In general, it concluded that students perform adequately when given investigatory problems in such an environment. They learn new mathematics and learn to handle the DGE at the same time.

1. Introduction

The fascinating discovery, namely that of the collinearity of the orthocentre, circumcentre and the centroid of a triangle was made by Leonard Euler in the 18th century. This result, famously known as the Euler line, is regarded as one of the finest results in geometry. As Sandifer [19] remarks

> *At the end of the nineteenth century, triangle geometry was regarded as one of the crowning achievements of mathematics, and the Euler line was one of its finest jewels.*

In a paper, published in 1767, Euler details out the analysis of the triangle centres and the relationships between them. However his main attempt was to solve a different problem – that of constructing a triangle, *given* the triangle centres. He used Heron's formula, elementary trigonometry (law of cosines) and the theory of equations to solve the problem and in the process made the observation about the collinearity of the three centres. Although he did not seem to consider this to be particularly important, it has indeed become one of the most celebrated results in geometry.

The geometrical proof of the collinearity of the orthocentre, circumcentre and the centroid of a triangle and the existence of the Euler line is well within the reach of high school students. School geometry is usually restricted to the study of lines, angles, triangles, quadrilaterals, circles and their properties. However, geometrical relations such as the Euler line are considered to be beyond the scope of the curriculum. The availability of dynamic geometry software, however, makes it possible for students to explore such results in a dynamic environment. In fact the use of Dynamic Geometry Environments (DGE) and their contribution to mathematics learning has been widely researched in the last two decades [3, 8, 10, 12 and16]. These studies have shown that dynamic geometry software, such as, Geometer's Sketchpad, GeoGebra and others, support creative discoveries and mathematical generalizations when used in a classroom. Many studies such as [4] have also tried to address the cognitive transition from experimental mathematics, that is, verification and conjecturing, to theoretical mathematics, that is, formal abstract concepts and proof.

In many countries across the world, such as India and Sweden, teachers do not traditionally use computers to demonstrate mathematical concepts or engage students in mathematical investigations. This is primarily because the curricula in these countries do not prescribe the use of computer technology for mathematics teaching and learning. Though schools and teachers have the freedom to use technology in their classrooms, their hesitation to do so may be attributed to lack of proper orientation and training. Further, in general, students are not given access to technology during examinations. Hence, there appears to be a lack of motivation to integrate technology in mathematics teaching in these countries.

1.1 Purpose of the study

In this article we shall describe how grade 9 students from two countries, India and Sweden, explored triangle centres, made conjectures regarding their collinearity and went on to developing a proof while working in a dynamic geometry environment. The study was designed to investigate the role of specific dynamic geometry software, GeoGebra, in enabling the students to explore the triangle centres. Although, culturally the two countries are different, there are similarities between the educational backgrounds of the students. In both countries, mathematics is largely taught as an abstract subject in the traditional 'chalk and board' manner, with little emphasis on applications. Also in the large majority of schools across India and Sweden, technology is hardly used for mathematics instruction. Even if technology is used, it is primarily for demonstration purposes and does not engage the students actively. Geometrical experiences of students in grades 9 in both India and Sweden, include properties of triangles, quadrilaterals, polygons, compass and ruler constructions, concepts of symmetry and congruency. The nature of the tasks posed to students at this stage is largely based on calculations and properties of figures. The notions of argumentation and proof are also introduced to students in grade 9, where they begin to reason about shapes using defined quantities and formulae.

Thus, we find that the approaches in mathematics teaching and learning in the two countries are similar in terms of emphasis on paper-pencil tasks and topics studied in geometry. Hence the authors (the researchers in this study) felt it would be worthwhile to see how grade 9 students of both countries, respond to using a technology tool in exploring a geometrical relations such as the Euler line. Teachers of both countries who participated in the study, exchanged notes regarding their students' performance in mathematics, to ensure that the students are of the similar levels of ability in terms of problem solving and conceptual understanding. Students in both countries underwent three two – hour sessions, in which they explored tasks based on three triangle centres and recorded their observations on worksheets. The students in both countries had no prior experience of working with a DGE.

1.2 Research questions

The objective of the study was to address the following research questions:

1. How do students perceive geometrical constructions in a dynamic geometry environment vis-à-vis a paper pencil environment?

2. How does a DGE help or hinder the process of making conjectures?
3. How do students to make the transition from conjecture to formal proof in a DGE? What is the nature of facilitation required to enable the transition?
4. Does the DGE act as an amplifier or a reorganiser or both?

2. Theoretical framework

2.1 Learning geometry through dynamic visual representations

Within the domain of mathematics, geometry is the study of shapes and space [20]. In the classroom, geometry is taught through a variety of representations, such as diagrams, schemes, drawings, and graphs. These representations are a contextual description of geometrical concepts or ideas and may support the process of conceptualization [21]. That is, the use of multiple representations facilitates student's development of geometrical concepts. Traditionally, geometry is taught and learned in a pencil and paper environment and geometry textbooks at schools provide iconic illustrations.

A conceptual understanding of geometry however requires mental imagination, since the proofs and derivations of formulas are based on flexibility and generalization of figures. Textbooks are not able to highlight the dynamic nature of geometrical figures. One may say there is a 'dynamic void' when geometry is learned through figures and diagrams in a textbook. Thus students are required to mentally investigate properties of geometrical objects without an external way to understand the related concepts, and as a result often fail to learn important concepts.

A Dynamic Geometry Environment provides a dynamic opportunity to the learning of geometry. It allows the user to perform investigations and thus affords the possibility of dynamic and visual representation of geometry concepts in a physical sense. Such investigatory activities are hard to experience in a static environment such as paper and pencil [8]. In Laborde's view, drawing refers to the material entity, while figuring refers to a theoretical object [5]. This distinction between drawing and figuring can be viewed as:

1. Some properties of a drawing might be irrelevant. If for example, a rhombus has been drawn as an instance of a parallelogram, then the equality of the sides is irrelevant.
2. The elements of the figure have an inbuilt variability that is absent in the drawing. For example, a parallelogram has infinitely many drawings; some of them are squares, some of them are rhombuses, and some of them are rectangles.
3. A single drawing may represent different figures. For example, a drawing of a square might represent a square, a rectangle, a rhombus, a parallelogram, or a quadrilateral.

Obviously, is not possible to provide an adequate representation for all properties simultaneously in a pencil and paper environment. Fortunately, this is possible with a DGE. According to Marrades and Gutiérrez [15]

> DGS helps teachers create learning environments where students can experiment, observe the permanence, or lack of permanence, of mathematical properties, and state or verify conjectures much more easily than in other computational environments or in the more traditional setting of paper and pencil. The main advantage of DGS learning environments over other (computational or non-computational) environments is that students can construct complex figures and can easily perform in real time a very wide range of transformations on those figures, so students have access to a variety of examples that can hardly be matched by non-computational or static computational environments (p.96)

Researchers have been particularly interested in student's understanding of geometrical concepts and objects and in investigating how they react when these geometrical concepts and objects are represented in a dynamical geometry system.

2.2 Theory of Variation: Dragging as a tool in a DGE

An individual may require several inputs (other than a definition) to help form a mental image of a geometrical concept. For example, to help an individual understand the idea of a triangle, the teacher may present several drawings or pictures of triangles, expecting the learner to 'see' some common features, among all the figures, namely, three sides, three vertices etc. The same idea may be explored in a dynamic geometry environment by drawing a triangle and dragging one of the vertices to create an animation, a continuous process, through which the learner will be able to see the invariant properties. Extending this a little further, we may want the learner to conjecture that the sum of the three angles of a triangle is 180^o. Presenting several triangles on paper and making them measure the three angles could be one way. However, in a dynamic geometry environment, the learner can drag the vertices of a triangle and 'see' (in the algebra view), that the sum remains 180^o irrespective of the shape of the triangle. Thus, the dragging tool is perhaps the most powerful feature of a DGE as it allows the user to abstract an idea by observing properties of figures, which remain invariant during the process of variation. According to Leung [11]:

> what makes a (DGE) a powerful mathematical knowledge acquisition micro world is its ability to visually make explicit the implicit dynamism of ''thinking about'' mathematical (in particular geometrical) concepts. By implicit dynamism I mean that when engaging in mathematical activities or reasoning, we often try to comprehend abstract concepts by some kind of ''mental animation''; that is, mentally visualizing variations of conceptual objects in the hope of ''seeing'' patterns of variation or invariant properties. The success of perceiving such patterns or properties often helps to bring about understanding of the underlying abstract mathematical concept.

Dragging in DGE is a much studied theme and plenty of research has been carried out to explore how dragging in DGE can be instrumental in helping students construct figures using their properties, explore geometrical problems, formulate conjectures and even proofs. Different dragging strategies or modalities have been investigated attempting to discover the roles that dragging can play in different DGE experiences [6, 7, 8, 9]. In particular, Arzarello et al. [1] identified seven dragging modalities (wandering, guided, bound, dummy locus, line, linked, drag test) while trying to analyse conjecture-making episodes by students working on a geometrical problem. Dragging has also been studied under the instrumentation approach (Verillon & Rabardel [23]) where the drag tool is scrutinized under the instrumental genesis framework.

Leung [11] also refers to the Variation theory and observes that variation is the epistemic essence of the drag mode in DGE. In particular, a key purpose in most DGE dragging strategies is to discover invariant properties in the midst of varying components of a geometrical configuration. Conceptualization of invariant structures amidst changing phenomena is often regarded as a key sign of knowledge acquisition. These dragging strategies often involve other functionalities in DGE like measurement, hide/show and trace to accentuate the different patterns of variation that can bring about the emergence of geometrical patterns.

Marton [13], in his work on learning and awareness, advocated that variation and simultaneity play a fundamental role in discernment:

> As we always act in relation to situations as we see them, effective actions spring from effective ways of seeing. Seeing a situation in a certain way amounts to discerning those aspects which are critical for engaging in effective action and taking all of them into consideration (focusing on them) at the same time. In order to discern a certain aspect, one

> must have experienced variation in those aspects. There is no discernment without variation (Marton et al. 2004, p. 14)

Marton et al. [13] proposed four inter-related functions (or patterns) of variation. They are described as follows:

Contrast: ''… in order to experience something, a person must experience something else to compare it with.''

Generalisation: ''… in order to fully understand what ''three'' is, we must also experience varying appearances of ''three'',…''

Separation: ''In order to experience a certain aspect of something, and in order to separate this aspect from other aspects, it must vary while other aspects remain invariant.''

Fusion: ''If there are several critical aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously.'' (Marton et al. 2004, p. 16).

In this article, these four functions of variation will act as pillars in helping us to deconstruct and make sense of the grade 9 students' exploration of the Euler Line.

2.3 DGE as amplifier and reorganiser

A further issue when using a DGE is that of the DGE acting as an *amplifier* or a *reorganizer* of mental activity [17, 18]. When technology is used as an amplifier, it performs more efficiently tedious processes (that might be done by hand), such as computations or the generation of standard mathematical representations such as graphs. In this use of technology, what students do or think about, are not changed but can be accomplished with significantly less time and effort and with more accuracy. For example, the use of a scientific calculator for computations while solving problems can make students' work more efficient and free from basic arithmetic errors. However, their activity and thinking is generally unchanged by this use of the calculator. On the other hand, when technology is used as a reorganiser, it has the power to shift the focus of students' mathematical thinking. It supports looking for patterns, identifying invariances or making and testing conjectures. Students can focus on developing insight rather than on drawing and measuring objects.

2.4 Theory of Cognitive Unity and Rupture

One of the objectives of school mathematics is to enable students to engage in argumentation and to learn deductive proof. However, this poses a huge challenge since students often are unable to produce deductive proofs for geometry problems. Researchers have studied this aspect in great detail.

Fiallo and Gutiérrez [4] describe proof as

> any mathematical argumentation raised to convince oneself or others of the truth of a mathematical statement.

They also differentiate between empirical and deductive proofs

> Empirical proofs, characterized by showing that the conjecture is true only in one or a few examples taken from a larger set of examples and assuming that the conjecture is also true in all other examples in the set. Deductive proofs, consisting of chains of logical implications connecting the hypothesis to the statement of the conjecture, and characterised by the decontextualisation of the ideas presented.
>
> Fiallo and Gutiérrez [4] state that
>
> Conjecture-and-proof problems are a key element in learning to prove. Their solution is divided into two parts: first, to state a conjecture, and second, to prove that the conjecture is true. Several

> researchers have analysed the processes of solving these problems to understand the relationship between students' reasoning in both parts.

Boero et al. (1996) propose the theory of the cognitive unity to explain possible reasons for students' the success or failure in writing deductive proofs. According to them

> there is cognitive unity when the argumentations used to produce and validate a conjecture help to construct its deductive proof. Otherwise, there is a cognitive rupture.

In exploring a geometrical problem students make conjectures based on their observations and produce empirical arguments. In order to develop the empirical argument into deductive proof they need to "decontextualize the argumentations to transform them into abstract deductive argumentations." [4]

The theoretical frameworks described in this section will be used to analyse 9th grade student's responses to geometrical tasks and their reasoning in developing deductive proof.

3. Research Methodology

In this section, we describe the main components of the intervention done by the researchers. We will focus on the student's mathematical background, the content of the investigative task, the nature of scaffolding required and the nature of students' investigations and argumentations.

3.1 Student's background

3.1.1 The sample

Twenty-two grade 9 students from India and twenty-eight grade 9 students from Sweden with similar mathematical ability (measured in grades) and with no prior DGE experience were selected by the researchers to participate in the study. A convenience sampling was done but it was ensured that the students of both countries had similar educational background in terms of exposure to computer use and mathematical concepts. Because of the similarity in the educational backgrounds and geometry experience, both groups of students may be considered as a single population in this study. The Swedish grade 9 students were selected from three ordinary community compulsory schools in Gothenburg. No significant difference in performance could be observed between students from the three schools. The Indian students who participated in the study were selected from three schools in Mumbai. These schools follow the curriculum prescribed by the State Board, which does not prescribe the use of technology for teaching mathematics or permit its use in examinations. The students who participated in this study knew how to construct the perpendicular bisectors, altitudes and medians of a triangle using ruler and compass but hadn't explored the same using any geometry software.

In India the students were given access to GeoGebra in a computer laboratory while in Sweden the students brought their own laptops. Both groups underwent three two - hour sessions during which they explored geometrical tasks using GeoGebra. These sessions were facilitated by the researchers in the presence of the teachers. The subjects of both countries had no prior experience in using GeoGebra. During the first session, they were familiarised with the basic tools and were assigned the task of constructing a square. This was a 'warm-up' exercise and the reason for starting with a construction task was primarily to get them familiar with the difference between drawing and construction, as well as basic construction tools of GeoGebra such as, the point tool, line tool, circle tool, etc. During this process, the researchers in each country conducted a whole-class discussion with the students, occasionally asking questions, and facilitating their understanding from time to time. The construction task took up about 25 minutes of the first two-hour session. After the construction, when students seemed fairly comfortable with the basic tools of GeoGebra, a worksheet comprising a set of investigative tasks based on the triangle centres, was given to them.

The reason for choosing the Euler Line Investigation was students' familiarity with properties of triangles and the concepts of altitudes, medians and perpendicular bisectors. They had studied these in their regular classes and had also constructed these using compass and ruler. However they were not introduced to the triangle centres. The investigation task provided students with opportunities for exploring the centres in detail and in making conjectures regarding their interrelationships.

In both India and Sweden, the researcher gave general instructions to facilitate students' investigations whenever required but interventions were kept to a minimum. The teachers moved around and helped if students faced any difficulty in using the GeoGebra tools. They did not however prompt answers or 'give away' solutions. Students' activities with GeoGebra were filmed (without students). At the end of the sessions all students turned in their worksheets and the voluntary groups saved their screen cast movies on R's USB stick (in two schools).

3.2 The intervention and teaching unit

In this section, we describe the content of the intervention tasks, the facilitation provided by the researchers through a guided discovery approach as well as students' responses. The intervention may be divided into two parts – the square construction task and the Euler Line investigation.

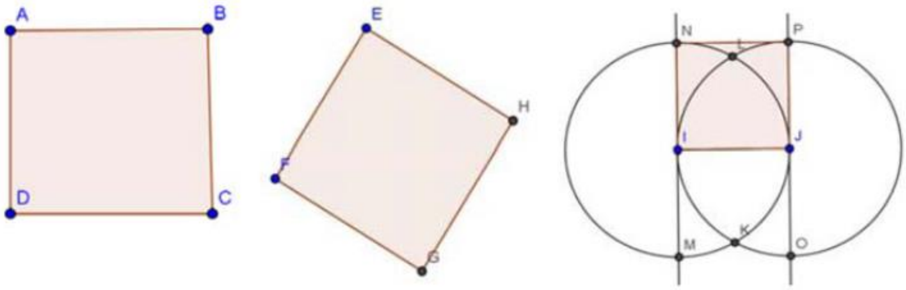
3.2.1 The square construction task

As a warm-up exercise, the researchers (in both countries) asked students to construct a square using GeoGebra. Other researchers [14, 22] have explored this classic construction problem, which requires geometric reasoning and familiarises the user with the construction tools of DGE. Initially most students attempted this task through manual adjustment. However, they soon realised that such a figure does not survive the 'drag test'. A few students discovered the regular polygon tool and used it to construct a square. However, others considered this as 'cheating' as it did not require the use of construction tools (point tool, line tool). This prompted the researchers to ask students to *construct* a square using GeoGebra's construction tools. At this point, students appeared to use their prior knowledge of constructing a square using a compass and ruler on paper to figure out the construction process in GeoGebra. They recalled that perpendiculars need to be drawn at end points A and B of a line segment AB. Following this, arcs are needed to ensure that the sides are of the same length. When asked why the arc was required, one student said, "the arc is needed to mark the point on the perpendicular line so that the sides of the square equal the line segment AB." They translated this process in GeoGebra and used the circle tool to complete the construction of the square (Figure 1). Any errors in the construction process were rectified using the hide/unhide or undo options. The GeoGebra construction required more mathematical knowledge about the properties of a square and circle than the paper-pencil construction.

One student observed, "when we construct a square using compass and ruler ... (like the way our teacher taught us), we cannot do anything to it afterwards. But this GeoGebra square can be moved and changed in both position and size and even after dragging it remains a square!" Here the student was comparing the static paper and pencil construction with the dynamic construction on GeoGebra. Dragging the vertices of the square made him realise that the side lengths and orientation of the square may vary but the equality of the sides and the right angles remains invariant. This illustrates the "drag for contrast approach" which enabled students to construct a 'robust' square that 'survived the drag test'. According to Leung [11]:

> Contrast is about seeing differences, comparing between what is and what is not, hence anticipating (conjecturing) what can be and cannot be (p. 6).

Further only after successfully constructing a robust square, which did not lose its properties upon dragging, did the students truly generalise the concept of the square. The "dragable" square in GeoGebra helped them to experience varying appearances of a square. Further, while dragging the vertices, they were able to separate out the fundamental properties of the square (equal sides and right angles) from the irrelevant properties (changing measure of sides and orientation of the square). Thus, in their visualization, the fundamental properties were brought to the fore while the irrelevant properties were pushed to the background. This illustrates *separation* as a function of variation. Finally the simultaneous experience of the varying and the invariant led to *fusion* and the complete visualisation of a square. Thus the four functions of variation -contrast, generalisation, separation and fusion played a key role in conceptualizing a square.



(a) (b) (c)

Figure 1: Students' constructions of a square: (a) by manual adjustment, (b) using the regular polygon tool (c) using line segment and circle tools (a robust construction).

3.2.2 The Euler Line Investigation

In this section, we shall describe the investigation task based on triangle centres and the Euler Line, which was attempted by students after the square construction task. Students were given a worksheet, which included the basic definitions of the centroid, circumcentre and orthocentre followed by a set of tasks (tasks 1 to 4 given below) for guiding the students through constructing the centres of a triangle.

Task 1: Construct a triangle ABC in GeoGebra. Construct the perpendicular bisectors. What do you observe? Mark the circumcentre of ABC as O and hide the perpendicular bisectors. Write your observations after dragging any one vertex of the triangle. Does the circumcentre always lie inside the triangle? What happens when the triangle is (i) acute, (ii) obtuse, (iii) or scalene?

Task 2: Construct the altitudes of triangle ABC. What do you observe? Mark the orthocentre of ABC as H and hide the altitudes. Write your observations after dragging any one vertex of the triangle. Does the orthocentre always lie inside the triangle? What happens when the triangle is (i) acute, (ii) obtuse, (iii) or scalene?

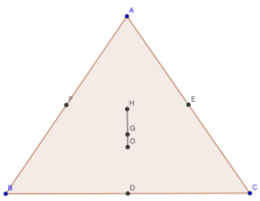
Task 3: Construct the medians of triangle ABC. What do you observe? Mark the centroid of ABC as G and hide the medians. Write your observations after dragging any one vertex of the triangle. Does the centroid always lie inside the triangle? What happens when the triangle is (i) acute, (ii) obtuse, (iii) or scalene?

Task 4: Explore the relationships between O, H and G, that is, the circumcentre, orthocentre and centroid of triangle ABC by dragging the vertices. What is your observation? Measure OG and HG each time you drag the vertices of the triangle. What do you observe about these measurements? If you have reached a conclusion, how certain are you that you are correct. Present your arguments.

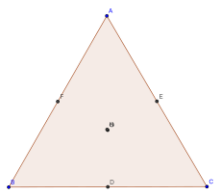
While attempting the first three tasks, about three – fourths of the students were able to construct the centres independently. The remaining students needed help to do the constructions on their respective GeoGebra screens. The teachers helped out with these students. After constructing each centre, students used the wandering dragging approach to make inferences as to whether these centres remained inside or outside the triangle. The variation caused by dragging the vertices of the triangle (making it acute, right, or obtuse) helped them to experience varying types of triangles and led them to make inferences regarding the location of the centres in each type of triangle. The simultaneous experience of the changing triangle and moving centres helped to arrive at conjectures. We see this as an example of *fusion*. Some argued, "the centroid is always inside…whether the triangle is acute, obtuse or right angled" and another observed, "the circumcentre remains inside only for an acute angled triangle". "In a right angled triangle the orthocentre is the vertex of the right angle" etcetera.

Once the three centres were constructed, task 4 appeared to proceed smoothly and was completed by all students. As students worked with all three centres simultaneously, they soon conjectured that these are collinear and constructed the Euler line. While dragging, they were able to separate out the collinearity of the three centres from other properties such as type of triangle or position of centre etc. We consider this ability to discern the collinearity of the three centres irrespective of the nature of the triangle as an example of *separation*. Thus, the variation experience caused by the dragging tool played a significant role in enabling students' conjecture making regarding the Euler Line.

After constructing the Euler's Line, the researchers urged the students to explore what happens in specific cases, such as, if the triangle is equilateral, isosceles or scalene. Interestingly, at this point, about two thirds of the students were able to use the guided dragging approach to manipulate the vertices of the triangle to make it an equilateral triangle (or isosceles). The rest needed a bit of facilitation (in terms of dragging to make a specific kind of triangle) which was provided by the teachers as and when needed. Figure 2 shows some GeoGebra constructions generated by students.



(a)



(b)

Figure 2. Students' investigation of Euler's line for (a) an Isosceles triangle and (b) an equilateral triangle.

The dragging tool enabled students to shift their focus from specific properties of the construction to general properties (such as collinearity of the centres leading to the Euler Line) and again from general to the specific (nature of the Euler line for specific types of triangles). Finally, after making conjectures related to the collinearity of the three centres leading to the Euler Line researchers scaffolded the students to prove their conjecture.

4. Results: An analysis of students' investigations

In this section, we present a detailed analysis of students' investigations as they explored the triangle centres leading to the emergence of the Euler Line.

4.1 DGE as an amplifier and reorganiser

Research question 4 attempts to address the role of DGE as an amplifier or a reorganiser in students' explorations of geometry tasks. In the Euler Line task, GeoGebra was the primary vehicle of exploration. Firstly, it served the purpose of a construction tool as students constructed the triangle and its centres. The constructions were accurate and it saved an enormous amount of time as performing the same constructions using a compass and ruler would have been very time consuming. Here the role of GeoGebra may be identified as that of an amplifier. However as students explored the rest of the task, making conjectures about the centres through dragging, the use of GeoGebra changed for the students. It shifted the students' focus from construction process to that of looking for invariances, making and testing conjectures. GeoGebra thus also played the role of a reorganiser. The following conversation among three Swedish students reveal how they negotiated the capabilities of GeoGebra to make constructions, arrive at conjectures and communicate their findings.

Student 1: *Gee, this is hard. I do not understand the English here. This is what it says. The circumcentre is the centre of the circumcircle of a triangle. Hey, please check Google translate about what the circumcentre means. And what on earth is a circumcircle? What are we expected to do?*

Student 2: *Right, here it is. You should first construct the midpoint of each side....and then construct a perpendicular line to that point and side. Look here it is...*

Student 1: *Okay, then what?*

Student 2: *Do it all the way around, do it for all sides of the triangle. Right, do you see that they all are intersecting at the same point? Here! Right click on that and label it O. This is apparently the midpoint for a circle. The other point for that circle is any of the vertices. Click over there... Yes! Cool! Let's make a screen cast of that!*

After some trial and error attempts, they manage to make a construction fairly quickly and efficiently. Here we may say that GeoGebra is playing the role of an amplifier. Student 3 who has been silent so far observes:

Student 3: *It seems that whatever we do to that triangle, the circle still exists. It means that every triangle always has a circumcircle even when we do not see it. It means that for every triangle you can think of or see, there are circles outside and inside and points of concurrency inside. It is so cool. I have never viewed triangles like this before.*

It appears that Student 3 is developing a sense of certain attributes of a triangle and is also observing invariances such as the existence of a circumcircle and points of concurrency. The dragging feature leads her to observe invariances amidst variation.

Student 3: *Look, if there is a circumcircle, there should be an inner circle to a triangle too? Right?*

Student 1: *Maybe, but it is not in the task.*

Student 3: *Look, here... I found it on the Internet. We should just divide the vertices angles in half and draw the segment to the opposite side. Let's do it, please!*

Student 3 convinces Student 1 and Student 2 that it will be worth the extra minutes and she guides them to do the construction. There is a moment when they do not agree about the circumference point.

Student 1: *Look, it is not inscribed because it crosses the triangle. There is something wrong.*

Student 3: *Yes, I got the wrong point. Look, you should use the perpendicular line from the centre point to each side and mark that intersection. Right... Now try again! Yes, there it is! It is so beautiful!*

Student 3 is affected by the richness of the triangle and she makes Student 1 colour it in a flashy way ensuring that her conjecture is proven.

Student 2: *Can we please go on and construct the Euler line and do the measurements now, please?*

Student 1: *Yes, and let's do a last screen cast as well.*

Finally the group constructed segments and measured the ratio between the distances GO and HG, and succeeded in describing the relationship among the three centres as HG = 2OG. This was an empirical proof created by students based on their observations.

This exploration would have been difficult to perform using compass and ruler as measurements are inaccurate and the process is also time consuming. The students revealed that they had a very positive impression of exploring the constructions in GeoGebra.

It would be apt to say that the Euler Line exploration in GeoGebra led to learning of mathematical concepts outside the activity and opened up new possibilities. We may conclude that initially GeoGebra acted as an amplifier as students used it primarily for performing constructions but later it played the role of a reorganiser by enabling students to make conjectures and giving them access to higher-level concepts.

Research question 2 attempts to explore the role played by a DGE in enabling the process of conjecture-making. The researcher (R) working with the Indian students observed that the dragging feature of GeoGebra helped students to make conjectures, which they might not have made in a paper-pencil environment. Before working on the Euler line investigation each pair of students was required to construct triangle of given specifications using compass and ruler. Following this they had to construct the three centres of their triangle. This was done so that they could later compare the compass and ruler method of construction with the GeoGebra construction. R took rounds to help students with the construction process as and when needed. This part was time consuming since students faced difficulties in constructing all the centres in one diagram. Once they identified the three centres, they marked them on their diagram. Each pair of students had a different triangle. Some were acute angled scalene triangles, some were obtuse angled, and some right angled, isosceles and equilateral. Students were unable to make any conjecture simply by looking at their own triangles. They began to look at their neighbour's triangles. Some did not get to see the circumcentre and orthocentre inside their triangle.

Student: *Teacher, in my triangle the circumcentre and orthocentre are not in the triangle. But in their triangle (he points to the neighbour) both lie inside.*

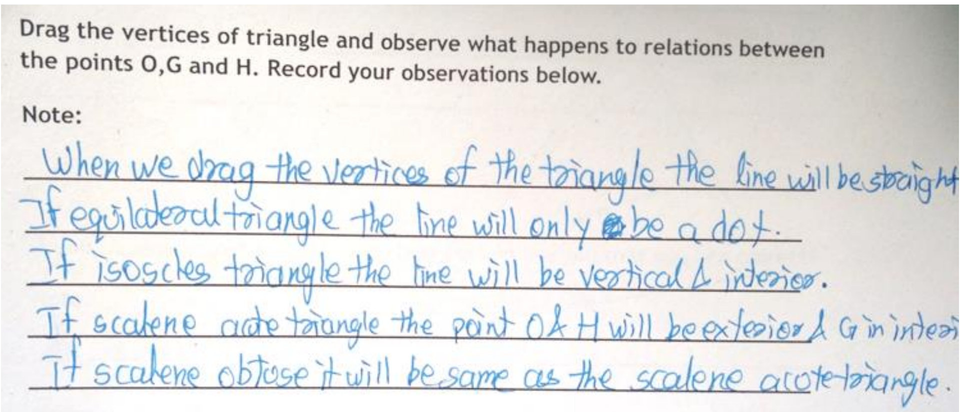
R: *Can you explain why? Is your triangle different from theirs?*

This activity led to interesting discussions but no consensus was reached regarding what kind of triangles had the circumcentres or orthocentres inside or outside. Some students who had made accurate constructions found the three centres lying on a straight line.

After this students attempted the worksheet tasks on GeoGebra. They followed the instructions given in the worksheet, which ultimately led them to construct the circumcentre, followed by the orthocentre and the centroid.

4.2 Conjecture through dragging

As articulated in research question 1, the researchers wanted to know how students perceived GeoGebra constructions vis-à-vis paper and pencil constructions. A majority of the students felt that GeoGebra constructions were easier, more accurate and less time consuming than compass and ruler constructions. There were 18 out of 22 students able to conclude that the three perpendicular bisectors and the three altitudes were concurrent while all students were convinced about the concurrency of the three medians. A majority of the students were able to conjecture that the circumcentre lies inside, outside or on the hypotenuse if the triangle was acute angled, obtuse angled or right angled respectively. Similarly about two thirds of the students could make similar conjectures regarding the orthocentre. In fact, they used wandering dragging as well as guided dragging approaches to discern invariances amidst variation and thus arrive at conjectures. There were 21 students conjectured, after dragging the vertices of the triangle, that the three centres are collinear. Using the guided dragging approach, 10 of these students observed that in an equilateral triangle the three centres coincide, in an isosceles triangle these lies on a vertical line and in a scalene triangle it is difficult to say. Figure 3 shows the conclusions written by a student in her worksheet. This is an example of an *empirical proof* as described by Fiallo and Gutiérrez [4], where she describes the relation between the three centres.


Drag the vertices of triangle and observe what happens to relations between the points O,G and H. Record your observations below.

Note:

When we drag the vertices of the triangle the line will be straight

If equilateral triangle the line will only be a dot.

If isoscles triangle the line will be vertical & interior.

If scalene acute triangle the point O & H will be exterior & G in interi

If scalene obtuse it will be same as the scalene acute triangle.

Figure 3: A student investigates the Euler's line when the triangle is equilateral, isosceles or scalene.

We would like to highlight here that such conjectures were not possible when the same task was attempted using compass and ruler. The explanation possibly lies in the absence of variation in this approach. Each pair of students had only one triangle and could make observations only based on it. However, there was no contrasting situation for them to observe.

4.3 Empirical to deductive proof

In this section, we shall try to elicit the students' thinking as they transitioned from empirical to deductive proof as we attempt to address research question 3.

4.3.1 Search for a conjecture and empirical proof

In the GeoGebra task, about two thirds of the students were able to describe the relationship among the three centres as HG = 2 OG. The measurements of line segments in the Algebra view helped to arrive at this relation and develop an empirical proof. This would have been difficult to verify and conjecture using compass and ruler construction because of the inaccuracy of measurements and cluttered diagrams. Initially GeoGebra acted as an amplifier by enabling students to make quick and accurate constructions and later it functioned as a reorganiser as it facilitated their conjecture making. The comments of the students revealed that students had a very positive impression of exploring the problem in GeoGebra. Some of them also expressed the desire of undergoing similar modules in other similar exploratory tasks. Figure 4 shows comments written by a student.

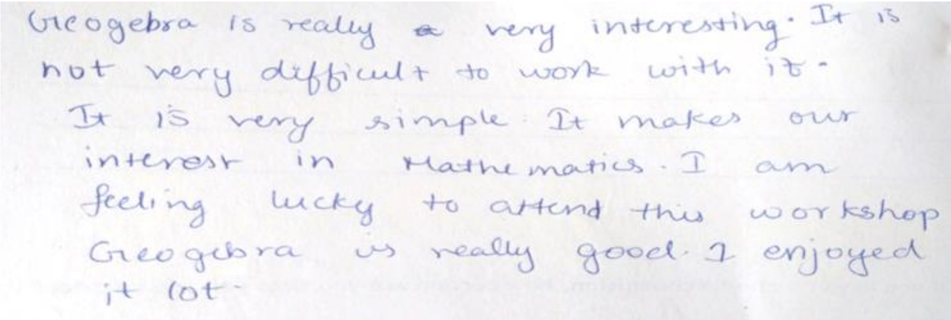


Figure 4: Students' comment on the Euler line exploration reveal their positive attitude towards such explorations.

At the end of the module students were required to develop a deductive proof for their conjecture regarding the collinearity of O, H and G. It was emphasized that the GeoGebra only helped to verify that the collinearity of the three centres and that this was not a deductive proof. The researcher tried to facilitate the process of developing a deductive proof and asked students to modify their GeoGebra constructions to obtain the following figure on their respective screens. See Figure 5.

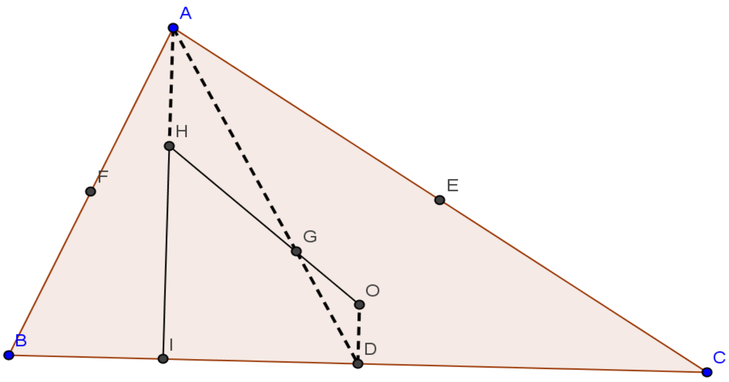
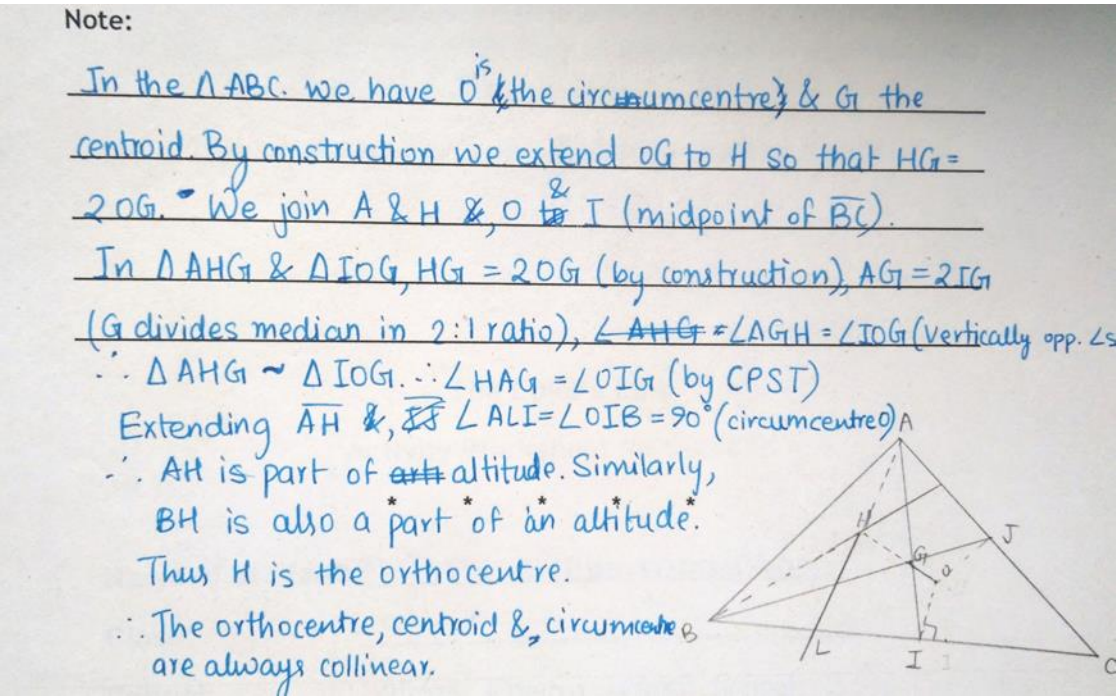


Figure 5: GeoGebra figure used by Indian 9th graders to prove collinearity of O,G and H.

4.3.2 Construction of deductive proof

The line of proof was facilitated through a guided discovery approach. The researcher conducted a whole class discussion where they drew students' attention to appropriate parts and properties of the figure, which were to be focused upon to develop the correct line of argument. An attempt was made to elicit students' reasoning and scaffold them to develop a chain of arguments without giving away any answers. In order to prove the collinearity of the three centres, the circumcentre O and the centroid G of a triangle ABC were constructed. OG was then extended to a point H such that HG = 2 OG. It was now required to prove that H is indeed the orthocentre of triangle ABC. This led to a discussion among students and to steer the discussion the researcher asked them to focus on the triangles AHG and OGD. They observed that AG = 2 GD, since G is the centroid and divided the median AD in the ratio 2:1. Also HG = 2 OG by construction. They also concluded that angles ADH and OGD are equal (being vertically opposite angles). These students were not familiar with the idea of similarity although they had learnt about the SAS criterion of congruency of triangles. It was reinforced by the researcher that SAS is also a criteria of similarity of triangles and that all the sides of triangle AHG are twice the lengths of the corresponding sides of triangle OGD. They further concluded that the corresponding angles of AHG and ODG are equal (being angles of similar triangles). This implied that AH is parallel to OD. Since OD is perpendicular to BC, it implied that AH, when extended is also perpendicular to BC and is therefore an altitude. Students further observed that dragging the vertices of triangle ABC did not alter the similarity of the triangles AHG and ODG. We see this as an example of *separation* as they were able to separate out the similarity of the triangles from other varying properties. The argument was extended by identifying the two triangles BHG and EOG and proving their similarity, which, finally led to the conclusion that BH when extended is also an altitude. Following the discussions students were required to write the formal proof in their worksheets. Figure 6 shows the proof written by a student in her worksheet.


Note:

In the ΔABC we have O is the circumcentre & G the centroid. By construction we extend OG to H so that HG = 2OG. We join A & H & O & I (midpoint of $\overline{BC}$).

In ΔAHG & ΔIOG, HG = 2OG (by construction), AG = 2IG (G divides median in 2:1 ratio), $\cancel{\angle AHG} = \angle AGH = \angle IOG$ (vertically opp. ∠s

∴ $\Delta AHG \sim \Delta IOG$. ∴ $\angle HAG = \angle OIG$ (by CPST)

Extending $\overline{AH}$ &, $\angle ALI = \angle OIB = 90°$ (circumcentre O)

∴ AH is part of altitude. Similarly, BH is also a part of an altitude.

Thus H is the orthocentre.

∴ The orthocentre, centroid & circumcentre are always collinear.

Figure 6: Students' written proof of the collinearity of the circumcentre, orthocentre and centroid of a triangle.

Here we can see that the empirical proof of the conjecture (of collinearity of the three centres) was a stepping-stone to developing a deductive proof. However, as described above, developing the deductive proof required facilitation by the researchers.

5. Discussion

There is a vast amount of published research on the use of DGE in teaching and learning mathematics. In this article, we have focused on analysing grade 9 students' explorations of a task in geometry form three perspectives. Firstly we have highlighted the dual role played by a DGE as an amplifier and reorganiser (research question 4), in supporting students' geometrical investigations, in providing a variation experience (through the dragging tool) for making conjectures (research question 2) and finally in motivating argumentation and proof. Secondly, we have tried to do an in-depth analysis of students' reasoning through the lens of the theory of variation. We feel that the four functions of variation, namely, contrast, generalisation, separation and fusion played a critical role in students' conjecture-making and developing proof. Thirdly, we describe students' transition in thinking as they progress from an empirical to a deductive proof (research question 3).

Students' perceptions suggest that GeoGebra offered many advantages in performing geometrical tasks over a paper-pencil environment. Firstly GeoGebra played the role of an *amplifier* in this study.

Students performed constructions quickly and they considered their GeoGebra constructions to be more accurate than compass and ruler constructions. Also when doing constructions, most students frequently used the algebra view to observe the properties of a given figure such as side lengths, angle measures, etc. Once the constructions were done, they moved away from measurements and focused on observing patterns, commonalities and invariances. Here GeoGebra acted as a *reorganiser* and played a key role in enabling students to make conjectures. We observed that for the 9th graders GeoGebra became a natural medium for their dynamical representations and facilitated their explorations helping them to identify invariant properties while performing the tasks. The ability to simultaneously see the changing measurements in the algebra view along with the change in the figure is a very powerful feature of a DGE, which has no competing counterpart in a paper-pencil environment. According to Leung [9]

> DGE is rooted in variation in its design. It is a milieu where mathematical concepts can be given dynamic visual forms subject to our action.

As a reorganiser, GeoGebra offered many possibilities throughout the task. It created a natural context for giving students access to higher-level mathematical concepts. For example, in an earlier section we saw how the Swedish students tried to explore the concept of incircle although it was not an assigned task. While attempting to prove the collinearity of the three centres, the Indian students were introduced to the idea of similarity of triangles, as an extension of their understanding of congruency.

In the square construction task, we saw evidence of all four functions of variation in enabling students to arrive at a robust square. The contrast between the manually adjusted square and a constructed square helped students to focus on and separate out the fundamental properties of the square. Dragging helped them to experience varying appearances of a square and generalise the concept.

In the Euler line exploration task, students used wandering dragging to conjecture whether the three centres remained inside the triangle or went outside for certain types of triangles. The simultaneous experience of the changing triangle and moving centres helped them to make conjectures about the location of centres for specific types of triangles. This is an example of *fusion*. Further, while dragging, they were able to discern the collinearity of the three centres irrespective of the nature of the triangle and this is an example of *separation*. After making the conjecture about the Euler Line, they used guided dragging to explore what happened to the Euler line for special cases such as an equilateral or an isosceles triangle.

Students agreed that figures drawn in a DGE are easier to draw, and once drawn they "react to the manipulations of the user following the laws of geometry". In fact, they were able to manipulate their GeoGebra figures (the hide and unhide features were frequently used) and also rectify errors easily. Thus, the transient nature of the DGE provided an additional advantage as a working tool vis-à-vis paper and pencil.

While proving the collinearity of the three centres, students first used their observations from the algebra view to develop an empirical proof that HG = 2OG. However, to lead them from empirical to deductive proof, the researchers had to facilitate by helping students to create an appropriate figure in GeoGebra (such as Figure 5). This helped them to observe that the similarity of the two triangles GAH and GOD remained invariant even as the vertices of the triangle were dragged. Here *separation* as a function of variation played a key role in helping them visualise the similarity of the triangles.

The researchers were surprised to find that Swedish students who are generally not exposed to mathematical proof in grade 9, attempted to strive for correctness and rigor while exploring the triangle centres. In various instances they engaged in argumentation and reasoning even when they were not prompted to do so. This supports the observation by Bruckheimer & Arcavi [2]

> While dynamic geometry software cannot actually produce proofs, the experimental evidence it provides users with produces strong conviction, which can motivate the desire for proof. (King & Schattschneider, 1997, p. xiii)

Thus, while conjectures and empirical proofs in GeoGebra set the stage for deductive proof, when it came to actually developing a deductive proof, students needed scaffolding. The researchers had to prompt students to create a chain of logical arguments, which would ultimately lead to proof. The nature of facilitation provided was a whole class discussion in which a guided discovery approach was followed wherein students were given the appropriate prompts to develop a logical sequence of arguments.

This kind of work, namely, experimentation with geometrical constructions leading to deductive proof, is probably something very rare and unusual in both India and Sweden at the compulsory school level. Nevertheless, students of both countries attempted the tasks with ease in GeoGebra and arrived at similar conjectures. Both Indian as well as Swedish students seemed to appreciate their mathematical experience during the study and the researchers were almost stunned by the speed with which they students picked up the constructions possibilities in GeoGebra. Our findings actually show that you most likely can take average students in compulsory school and ask them to do geometrical constructions they haven't seen before and they will do it, and they most likely will make conjectures and draw conclusions along the way.

Further, the researchers in both countries observed that the process of developing deductive proof from an empirical one became a natural transition for the 9th graders. The relevant factors facilitating this transition were the DGS environment, which helped students to discover and verify conjectures, as well as the teaching methodology and facilitation, which promoted discussion and asked students for justifications of their arguments. In the process of writing the proof, however, GeoGebra's role was restricted. Here they converted their observations/verifications (made using GeoGebra) to a sequence of logical arguments, which then led to the final proof. In the process of writing the proof, some facilitation was provided by the researchers.

The process of proving the collinearity of the three centres of a triangle led to the proof similarity of triangles. This enabled the researcher to facilitate a discussion on similarity of triangles and help students figure out the different criteria for similarity. Typically the topic of similarity of triangles is a part of the curriculum of grade 10 and hence the exploration of triangle centres familiarised students with concepts beyond their traditional syllabus. Here too the researchers observed that the DGE played an important role of a reorganiser in shaping their concept of similarity as it quickened the process of making and verifying conjectures. However when it came to formal argumentation and proof (such as proving that H is the orthocentre or that the nine points lie on a circle) students referred to their GeoGebra figures but wrote their arguments on paper with some facilitation by the researcher.

The interaction with students reveals that the DGE helped to make and verify conjectures. It also led to a 'need for proof' of different results, which students were encountering for the first time. The DGE appeared to provide a motivation for posing formal arguments leading to a proof, which in general seems to be missing in traditional classrooms. Thus the process of 'finding a proof' seemed to be a natural extension of the process of making conjectures.

In the Results section we observed how Student 3 developed a much broader image of what a triangle is or might be and perhaps experienced an 'ontological shift', a sudden ability to see a familiar object, the triangle, in a new light and with completely new qualities. Actually it moved her to deepen the investigation. Once again GeoGebra played the role of an amplifier.

Thus we may conclude that throughout the Euler Line exploration GeoGebra played the role of an amplifier as well as a reorganiser. In the initial tasks, students focused on constructions (medians, altitudes and perpendicular bisectors) and identified the centroid, orthocentre and the circumcentre of the triangle. These constructions were performed quickly, were accurate and dynamic in nature. During this process GeoGebra played the role of an amplifier. After this, students moved away from the construction process and began to look for patterns, focus on invariances and make conjectures. Here GeoGebra played the role of a reorganiser by increasing the possibilities for exploration.

As we analyse students' explorations and reasoning closely, we see evidence of the four functions of variation. Wandering dragging and guided dragging led students to make conjectures. By the end of the third session of this activity, we had witnessed a panoramic view of students' thinking.

Finally it would be apt to say that the 9th grade students in both India and Sweden learned new geometry while taking part in this study. They also learned to use GeoGebra as a medium for exploring geometrical problems, which is likely to be beneficial in their future learning and study of mathematics. It may be appropriate to say that the use of GeoGebra led to a satisfying combination of technology use and engagement with logical reasoning. The results of the study were illuminating for the teachers (of both countries whose students had participated in the study) as well as the researchers. The teachers realised that their students were quite comfortable with GeoGebra and the new dynamic environment enable students to do geometrical explorations far beyond the level of grade 9. The researchers were highly encouraged by the results as they could see how GeoGebra had empowered the students by giving them access to higher-level concepts while exploring geometrical problems and thus develop their mathematical thinking.

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